CR Model CF1 Sensitivity Engine

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Abstract

We build a self-contained, implementation-ready derivation of first-order sensitivities of $\mathrm{SCR}_\alpha = \mathrm{VaR}_\alpha - \mathbb{E}[\cdot]$ for aggregate insurance losses under the collective risk model. The development is distribution-agnostic up to third moments, supports a fixed cross-LoB correlation matrix, and adopts a Level-0 portfolio skewness closure (no cross-coskewness). We justify the third central moment of a compound sum via total cumulance and via cgf composition, derive the CF1 differential, lift it to a correlated portfolio, present alternative six-d.o.f. parameterizations, and provide iso-capital trade-offs and a waterfall-ready decomposition. An algorithmic section concludes with exact inputs, precomputations, and formulas to wire into a UI with sliders.

1 Model and notation

Assumption 1 (Collective risk model)**.** For a sub-LoB, total loss $S = \sum_{i=1}^{N} X_i$, with $N \perp \{X_i\}$ and $X_i \overset{iid}{\sim} X$. Moments exist up to order three.

Severity: $\mu_X = \mathbb{E}[X]$, $\sigma_X^2 = \mathrm{Var}(X)$, $\mu_{3,X} = \mathbb{E}[(X-\mu_X)^3]$, $\gamma_X = \mu_{3,X}/\sigma_X^3$.

Frequency: $\mu_N = \mathbb{E}[N]$, $\sigma_N^2 = \mathrm{Var}(N)$, $\mu_{3,N} = \mathbb{E}[(N-\mu_N)^3]$, $\gamma_N = \mu_{3,N}/\sigma_N^3$.

Aggregate: $\mu_S = \mathbb{E}[S]$, $v_S = \mathrm{Var}(S)$, $\sigma_S = \sqrt{v_S}$, $\mu_{3,S} = \mathbb{E}[(S-\mu_S)^3]$, $\gamma_S = \mu_{3,S}/\sigma_S^3$.

Capital: $\mathrm{SCR}_\alpha(Y) = \mathrm{VaR}_\alpha(Y) - \mathbb{E}[Y]$, with $z = z_\alpha = \Phi^{-1}(\alpha)$.

Portfolio: LoBs $k = 1, \ldots, K$ with totals S_k, standard deviations σ_k, stack $s = (\sigma_1, \ldots, \sigma_K)^\top$, and fixed correlation matrix $\mathbf{P} = (\rho_{ij})$.

2 Single-LoB moments and the CF1 differential

Proposition 1 (Cumulant map and third central moment)**.** *Under assumption 1,*

$$\mu_S = \mu_N \mu_X, \qquad v_S = \mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2, \tag{1}$$

and

$$\mu_{3,S} = \mu_N \mu_{3,X} + 3\sigma_N^2 \mu_X \sigma_X^2 + \mu_{3,N} \mu_X^3 = \mu_N \gamma_X \sigma_X^3 + 3\sigma_N^2 \mu_X \sigma_X^2 + \gamma_N \sigma_N^3 \mu_X^3. \tag{2}$$

Proof (law of total cumulance). For any Y and Z, $\kappa_3(Y) = \mathbb{E}[\kappa_3(Y|Z)] + \kappa_3(\mathbb{E}[Y|Z]) + 3\mathrm{Cov}(\mathbb{E}[Y|Z], \mathrm{Var}(Y|Z))$. Condition on N: $\mathbb{E}[S|N] = N\mu_X$, $\mathrm{Var}(S|N) = N\sigma_X^2$, $\kappa_3(S|N) = N\mu_{3,X}$. Substitute to obtain (2); (1) follows from the law of total variance and the assumptions of independence. □

Remark 1 (Alternative proof (cgf composition))**.** The cgf composition $K_S(t) = K_N(\log M_X(t))$ and Faà di Bruno yield $\kappa_3(S) = \kappa_1(N)\kappa_3(X) + 3\kappa_2(N)\kappa_1(X)\kappa_2(X) + \kappa_3(N)\kappa_1(X)^3$, which becomes (2) after identifying $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$, $\kappa_3 = \mu_3 = \gamma\sigma^3$.

Proposition 2 (CF1 capital and its differential)**.** *Cornish–Fisher (1-term) gives*

$$\mathrm{VaR}_\alpha(S) \approx \mu_S + \sigma_S\Big[z + \frac{\gamma_S}{6}(z^2-1)\Big]. \tag{3}$$

Thus

$$\mathrm{SCR}_\alpha(S) \approx \sigma_S\Big[z + \frac{\gamma_S}{6}(z^2-1)\Big]. \tag{4}$$

Its first-order total differential is

$$d\mathrm{SCR}_\alpha(S) = \underbrace{\frac{1}{2\sigma_S}\Big[z - \frac{\gamma_S}{3}(z^2-1)\Big]}_{c_1(S)}\, dv_S \;+\; \underbrace{\frac{z^2-1}{6\sigma_S^2}}_{c_2(S)}\, d\mu_{3,S}. \tag{5}$$

Proof. Differentiate (4) using $d\sigma_S = \frac{1}{2\sigma_S} dv_S$ and $d\gamma_S = \sigma_S^{-3} d\mu_{3,S} - 3\mu_{3,S}\sigma_S^{-4} d\sigma_S$. Collect terms to get (5). □

Chain rule (single LoB). From (1)–(2), with parameters $\theta = (\mu_X, \sigma_X^2, \gamma_X; \mu_N, \sigma_N^2, \gamma_N)$,

$$dv_S = 2\sigma_N^2\mu_X\, d\mu_X + \mu_N\, d\sigma_X^2 + \sigma_X^2\, d\mu_N + \mu_X^2\, d\sigma_N^2, \tag{6}$$

$$\begin{aligned} d\mu_{3,S} = {} & \big(3\sigma_N^2\sigma_X^2 + 3\gamma_N\sigma_N^3\mu_X^2\big)\, d\mu_X + \Big(\tfrac{3}{2}\mu_N\gamma_X\sigma_X + 3\sigma_N^2\mu_X\Big)\, d\sigma_X^2 + \mu_N\sigma_X^3\, d\gamma_X \\ & + \gamma_X\sigma_X^3\, d\mu_N + \Big(3\mu_X\sigma_X^2 + \tfrac{3}{2}\gamma_N\sigma_N\mu_X^3\Big)\, d\sigma_N^2 + \sigma_N^3\mu_X^3\, d\gamma_N. \end{aligned} \tag{7}$$

Define Jacobians $J_V = \nabla_\theta v_S$ and $J_{M3} = \nabla_\theta \mu_{3,S}$ read off from (6)–(7). Then $\nabla_\theta \mathrm{SCR}_\alpha(S) = c_1(S)\, J_V + c_2(S)\, J_{M3}$.

3 Correlation-aware portfolio lift (Level-0 coskew)

Lemma 1 (Portfolio variance with fixed correlation)**.** *Let $v_i = \sigma_i^2$ and $v_T = s^\top \mathbf{P} s$. Then*

$$dv_T = \sum_{i=1}^K m_i\, dv_i, \qquad m_i = \frac{(\mathbf{P}s)_i}{\sigma_i} = 1 + \sum_{j\neq i} \rho_{ij}\frac{\sigma_j}{\sigma_i}. \tag{8}$$

Proof. $dv_T = 2s^\top \mathbf{P}\, ds$ since $\mathbf{P}$ is symmetric. With $ds_i = \frac{1}{2\sigma_i} dv_i$, collect coefficients of each dv_i. □

Definition 1 (Level-0 portfolio skewness closure)**.** Ignore cross-coskewness and approximate

$$\mu_{3,T} \approx \sum_{i=1}^K \mu_{3,S_i}, \qquad \gamma_T = \mu_{3,T}/\sigma_T^3, \qquad \sigma_T = \sqrt{v_T}. \tag{9}$$

Proposition 3 (Portfolio CF1 differential (Level-0))**.** *Let $c_1(T) = \frac{1}{2\sigma_T}\big[z - \frac{\gamma_T}{3}(z^2-1)\big]$ and $c_2(T) = \frac{z^2-1}{6\sigma_T^2}$. For each LoB i with parameter block θ_i,*

$$d\mathrm{SCR}_\alpha(T) = c_1(T)\sum_{i=1}^K m_i\, \langle J_{V,i},\, d\theta_i\rangle + c_2(T)\sum_{i=1}^K \langle J_{M3,i},\, d\theta_i\rangle. \tag{10}$$

Remark 2 (Upgrades when available)**.** If calibrated pair/triple coskewness surrogates are introduced, replace $J_{M3,i}$ by $\widetilde{J}_{M3,i} = J_{M3,i} + (\text{coskewness terms})$; (8) and the variance side are unchanged.

4 Alternative six-d.o.f. parameterizations

Severity via coefficient of variation. Let $c_X = \sigma_X/\mu_X$, parameters (μ_X, c_X, γ_X). Map $\sigma_X^2 = (c_X\mu_X)^2$ with $\partial\sigma_X^2/\partial\mu_X = 2c_X^2\mu_X$, $\partial\sigma_X^2/\partial c_X = 2c_X\mu_X^2$. Apply the chain rule to $\nabla_\theta \mathrm{SCR}$.

Frequency via overdispersion. Let $\delta_N = (\sigma_N^2 - \mu_N)/\mu_N^2$ (Poisson $\delta_N = 0$), parameters $(\mu_N, \delta_N, \gamma_N)$. Map $\sigma_N^2 = \mu_N + \delta_N\mu_N^2$ with $\partial\sigma_N^2/\partial\mu_N = 1 + 2\delta_N\mu_N$, $\partial\sigma_N^2/\partial\delta_N = \mu_N^2$. Again apply the chain rule.

5 Iso-capital moves and a waterfall-ready decomposition

Gradient blocks. Per sub-LoB i,

$$g_i := c_1(T)\, m_i\, J_{V,i} + c_2(T)\, J_{M3,i} \in \mathbb{R}^6. \tag{11}$$

Free prediction (sliders). For proposed changes $\{\Delta\theta_i\}$,

$$\Delta\mathrm{SCR}_\alpha(T) \approx \sum_{i=1}^{K} \langle g_i,\, \Delta\theta_i\rangle. \tag{12}$$

Iso-capital trade-off (one adjuster). Choose adjuster block u and enforce $\Delta\mathrm{SCR}_\alpha(T) = 0$:

$$\Delta\theta_u = -\,\frac{\sum_{i\neq u}\langle g_i,\, \Delta\theta_i\rangle}{\|g_u\|^2}\, g_u. \tag{13}$$

Iso-capital trade-off (many adjusters). Let G_A stack the g_j for adjusters $j \in A$, then solve

$$(G_A^\top G_A)\,\Delta\theta_A = -\,G_A^\top \sum_{i\in D} g_i\,\Delta\theta_i. \tag{14}$$

Waterfall decomposition. Given observed change $\Delta\mathrm{SCR}^{\mathrm{obs}}$,

$$\Delta\mathrm{SCR}^{\mathrm{obs}} \approx \underbrace{c_1(T)\,\Delta v_T}_{\text{variance}} + \underbrace{c_2(T)\,\Delta\mu_{3,T}}_{\text{skewness}} + \underbrace{\text{Residual}}_{\text{CF truncation, nonlinearity, unmodeled dependence}}. \tag{15}$$

(Optionally add a CF2 bar $\delta_{\mathrm{CF2}} \approx \sigma_T\big[\frac{\kappa_T}{24}(z^3 - 3z) - \frac{\gamma_T^2}{36}(2z^3 - 5z)\big]$ if an estimate of κ_T is available.)

6 Implementation blueprint (plug-and-play)

Inputs at baseline

1. For each LoB i: $\mu_{X,i}, \sigma^2_{X,i}, \gamma_{X,i}$; $\mu_{N,i}, \sigma^2_{N,i}, \gamma_{N,i}$.
2. Correlation matrix $\mathbf{P}$ across LoB totals.
3. Confidence level α (compute $z = z_\alpha$).

Per-LoB precompute

1. $v_i = \mu_{N,i}\sigma^2_{X,i} + \sigma^2_{N,i}\mu^2_{X,i}$.
2. $\mu_{3,S_i} = \mu_{N,i}\gamma_{X,i}\sigma^3_{X,i} + 3\sigma^2_{N,i}\mu_{X,i}\sigma^2_{X,i} + \gamma_{N,i}\sigma^3_{N,i}\mu^3_{X,i}$.
3. Store $J_{V,i}$ from (6) and $J_{M3,i}$ from (7).

Portfolio aggregate

1. $s_i = \sqrt{v_i}$, $s = (s_1, \ldots, s_K)^\top$.
2. $v_T = s^\top \mathbf{P} s$, $\sigma_T = \sqrt{v_T}$.
3. $\mu_{3,T} = \sum_i \mu_{3,S_i}$ (Level-0), $\gamma_T = \mu_{3,T}/\sigma^3_T$.
4. $c_1(T) = \frac{1}{2\sigma_T}\big[z - \frac{\gamma_T}{3}(z^2-1)\big]$, $\quad c_2(T) = \frac{z^2-1}{6\sigma^2_T}$.
5. $m_i = \frac{(\mathbf{P}s)_i}{s_i} = 1 + \sum_{j\neq i}\rho_{ij}\frac{s_j}{s_i}$.
6. $g_i = c_1(T)\, m_i\, J_{V,i} + c_2(T)\, J_{M3,i}$.

Slider prediction and iso-capital

1. Predict $\Delta\mathrm{SCR}_\alpha(T)$ via (12).
2. Enforce iso-capital via (13) or (14).

Waterfall

1. Variance bar $= c_1(T)\sum_i m_i\, \Delta v_i$.
2. Skewness bar $= c_2(T)\sum_i \Delta\mu_{3,S_i}$.
3. Residual $= \Delta\mathrm{SCR}^{\mathrm{obs}} -$ Predicted (add CF2 bar if available).

Sanity checks and special cases

Poisson frequency. If $N \sim \mathrm{Pois}(\lambda)$, then $\sigma^2_N = \mu_N = \lambda$ and $\gamma_N = \lambda^{-1/2}$. Plug into (2) and all subsequent formulas.

Deterministic frequency. If $N \equiv n$, then $\sigma^2_N = \mu_{3,N} = 0$, so $\mu_{3,S} = n\,\mu_{3,X} + 3n(n-1)\mu_X\sigma^2_X + n(n-1)(n-2)\mu^3_X$.

Identifiability. Third raw moments are not determined by means and variances alone: $\mu_{3,X}$ and $\mu_{3,N}$ (or γ_X, γ_N) are needed.

Appendix: explicit Jacobians (single LoB)

Order parameters as $\theta = (\mu_X, \sigma^2_X, \gamma_X; \mu_N, \sigma^2_N, \gamma_N)$. Then

$$J_V = \begin{bmatrix} 2\sigma^2_N\mu_X \\ \mu_N \\ 0 \\ \sigma^2_X \\ \mu^2_X \\ 0 \end{bmatrix}, \qquad J_{M3} = \begin{bmatrix} 3\sigma^2_N\sigma^2_X + 3\gamma_N\sigma^3_N\mu^2_X \\ \frac{3}{2}\mu_N\gamma_X\sigma_X + 3\sigma^2_N\mu_X \\ \mu_N\sigma^3_X \\ \gamma_X\sigma^3_X \\ 3\mu_X\sigma^2_X + \frac{3}{2}\gamma_N\sigma_N\mu^3_X \\ \sigma^3_N\mu^3_X \end{bmatrix}.$$