Collective Risk Model with NB Freq and LN Seve: Derivations for $\mathrm{VaR}_{0.995}$, Skewness, and Sensitivities

Notation

All quantities are real and finite up to order 3.

- Random variables: X (severity), N (claim count), $S = \sum_{i=1}^{N} X_i$ (aggregate), with $N \perp X_i$ and $X_i \overset{\text{i.i.d.}}{\sim} X$.
- Means/variances/skewnesses: $\mu_X = \mathbb{E}[X]$, $\sigma_X^2 = \mathrm{Var}(X)$, $\gamma_X = \mathbb{E}[(X-\mu_X)^3]/\sigma_X^3$. Likewise $(\mu_N, \sigma_N^2, \gamma_N)$.
- For portfolio with K sub-LoBs, stack drivers as vectors

$$\boldsymbol{n} = \begin{bmatrix} N_{E,1} & \cdots & N_{E,K} \end{bmatrix}^\top,$$

$$\boldsymbol{o} = \begin{bmatrix} \mathrm{OD}_{N,1} & \cdots & \mathrm{OD}_{N,K} \end{bmatrix}^\top,$$

$$\boldsymbol{x} = \begin{bmatrix} X_{E,1} & \cdots & X_{E,K} \end{bmatrix}^\top,$$

$$\boldsymbol{a} = \begin{bmatrix} \mathrm{CV}_{X,1}^2 & \cdots & \mathrm{CV}_{X,K}^2 \end{bmatrix}^\top,$$

  where $\mathrm{OD}_N := N_V/N_E$ (≥ 1) and $\mathrm{CV}_X^2 := X_V/X_E^2$.
- Elementwise algebra: Hadamard product $\odot$, elementwise powers $\cdot^{\circ p}$, differentials in Fréchet form (no indices).
- Standard normal quantile: $z_\alpha = \Phi^{-1}(\alpha)$ (at $\alpha = 0.995$, $z_{0.995} \approx 2.5758293035$).

1 Skewness of a product (warm-up)

Let $Z = XY$ with $X \perp Y$. Using raw moments $a_k = \mathbb{E}[X^k]$ and $b_k = \mathbb{E}[Y^k]$,

$$\mathbb{E}[Z] = \mu_X \mu_Y, \qquad \mathrm{Var}(Z) = a_2 b_2 - \mu_X^2 \mu_Y^2, \tag{1}$$

$$\mu_3(Z) := \mathbb{E}[(Z - \mathbb{E}Z)^3] = a_3 b_3 - 3\,\mu_X \mu_Y\, a_2 b_2 + 2\,\mu_X^3 \mu_Y^3. \tag{2}$$

Hence the skewness is

$$\gamma_Z = \frac{a_3 b_3 - 3\,\mu_X \mu_Y\, a_2 b_2 + 2\,\mu_X^3 \mu_Y^3}{(a_2 b_2 - \mu_X^2 \mu_Y^2)^{3/2}}. \tag{3}$$

In (μ, σ, γ) form, $a_2 = \sigma_X^2 + \mu_X^2$, $a_3 = \mu_X^3 + 3\mu_X \sigma_X^2 + \gamma_X \sigma_X^3$, etc., which yields

$$\mu_3(Z) = \gamma_X \gamma_Y\, \sigma_X^3 \sigma_Y^3 + \gamma_X \sigma_X^3\, \mu_Y (3\sigma_Y^2 + \mu_Y^2) + \gamma_Y \sigma_Y^3\, \mu_X (3\sigma_X^2 + \mu_X^2) + 6\,\mu_X \mu_Y\, \sigma_X^2 \sigma_Y^2, \tag{4}$$

$$\mathrm{Var}(Z) = \sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \mu_X^2 \sigma_Y^2, \qquad \gamma_Z = \mu_3(Z)/\mathrm{Var}(Z)^{3/2}. \tag{5}$$

Special cases: if $\mu_X = \mu_Y = 0$ then $\gamma_Z = \gamma_X \gamma_Y$; if both are Gaussian (zero skew), only the $6\,\mu_X \mu_Y \sigma_X^2 \sigma_Y^2$ term remains in $\mu_3(Z)$.

2 Collective risk model: general identities

With $S = \sum_{i=1}^{N} X_i$ and $N \perp X$,

$$S_E := \mathbb{E}[S] = N_E\, X_E, \tag{6}$$

$$S_V := \mathrm{Var}(S) = N_E\, X_V + X_E^2\, N_V, \tag{7}$$

$$\mu_3(S) = N_E\, X_3 + 3\, N_V\, X_E X_V + N_3\, X_E^3, \tag{8}$$

$$\gamma_S = \mu_3(S)/S_V^{3/2}. \tag{9}$$

The differentials (no index notation) are

$$\mathrm{d}S_E = S_E\,(\,\mathrm{dln}\, N_E + \,\mathrm{dln}\, X_E), \tag{10}$$

$$\mathrm{d}S_V = X_V\,\mathrm{d}N_E + N_E\,\mathrm{d}X_V + 2X_E N_V\,\mathrm{d}X_E + X_E^2\,\mathrm{d}N_V. \tag{11}$$

3 Frequency: Negative Binomial; Severity: Lognormal

Closed-form component moments in exogenous drivers

Take drivers $(N_E, \mathrm{OD}_N, X_E, A)$ with $A := \mathrm{CV}_X^2$.

Negative Binomial (NB). With $N_V = N_E\,\mathrm{OD}_N$ and $r = N_E/(\mathrm{OD}_N - 1)$,

$$N_3 = N_E\,\mathrm{OD}_N\,(2\,\mathrm{OD}_N - 1), \qquad \gamma_N = \frac{2\,\mathrm{OD}_N - 1}{\sqrt{N_E\,\mathrm{OD}_N}}. \tag{12}$$

Lognormal (LN). Let $\sigma^2 = \ln(1 + A)$ and $\mu = \ln X_E - \frac{1}{2}\sigma^2$. Then

$$X_V = X_E^2\, A, \quad X_3 = X_E^3\, A^2(A + 3), \quad \gamma_X = (A + 3)\sqrt{A}. \tag{13}$$

Aggregate moments for one sub-LoB

Using (7)–(8),

$$S_E = N_E X_E, \tag{14}$$

$$S_V = N_E X_E^2\,(A + \mathrm{OD}_N), \tag{15}$$

$$\mu_3(S) = N_E X_E^3\Big[A^3 + 3A^2 + 3\,\mathrm{OD}_N\, A + 2\,\mathrm{OD}_N^2 - \mathrm{OD}_N\Big], \tag{16}$$

$$\gamma_S = \frac{A^3 + 3A^2 + 3\,\mathrm{OD}_N\, A + 2\,\mathrm{OD}_N^2 - \mathrm{OD}_N}{\sqrt{N_E}\,(A + \mathrm{OD}_N)^{3/2}}. \tag{17}$$

Checks: Poisson limit $\mathrm{OD}_N \to 1$ gives $\gamma_S = (A+1)^{3/2}/\sqrt{N_E}$; deterministic N ($\mathrm{OD}_N \to 0$) gives $\gamma_S = \gamma_X/\sqrt{N_E}$.

4 Cornish–Fisher q_α and SCR at $\alpha = 0.995$

Define

$$\Theta_\alpha := z_\alpha + \frac{\gamma_S}{6}\,(z_\alpha^2 - 1). \tag{18}$$

Then

$$\mathrm{VaR}_\alpha(S) \approx S_E + \sqrt{S_V}\,\Theta_\alpha, \tag{19}$$

$$\mathrm{SCR}_\alpha := \mathrm{VaR}_\alpha(S) - S_E \approx \sqrt{S_V}\,\Theta_\alpha. \tag{20}$$

At $\alpha = 0.995$, $z = z_{0.995} \approx 2.5758293035$ and with (17),

$$\boxed{\mathrm{SCR} = X_E \left[z \sqrt{N_E(A + \mathrm{OD}_N)} + \frac{z^2 - 1}{6} \, \frac{A^3 + 3A^2 + 3\,\mathrm{OD}_N\, A + 2\,\mathrm{OD}_N^2 - \mathrm{OD}_N}{\sqrt{N_E(A + \mathrm{OD}_N)}} \right].} \tag{21}$$

5 Portfolio of independent sub-LoBs: vector calculus

Stack $(\boldsymbol{n}, \boldsymbol{o}, \boldsymbol{x}, \boldsymbol{a}) \in \mathbb{R}^{4K}$. Elementwise,

$$\boldsymbol{s}_E = \boldsymbol{n} \odot \boldsymbol{x}, \tag{22}$$

$$\boldsymbol{s}_V = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 2} \odot (\boldsymbol{a} + \boldsymbol{o}), \qquad S_V = \mathbf{1}^\top \boldsymbol{s}_V, \tag{23}$$

$$\boldsymbol{m}_3 = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 3} \odot \Big(\boldsymbol{a}^{\circ 3} + 3\boldsymbol{a}^{\circ 2} + 3\boldsymbol{o} \odot \boldsymbol{a} + 2\boldsymbol{o}^{\circ 2} - \boldsymbol{o} \Big), \qquad \mu_3 = \mathbf{1}^\top \boldsymbol{m}_3, \tag{24}$$

$$S_E = \mathbf{1}^\top \boldsymbol{s}_E, \qquad \gamma_S = \mu_3 / S_V^{3/2}. \tag{25}$$

Hence

$$\mathrm{VaR}_\alpha(S) \approx S_E + \sqrt{S_V} \left(z_\alpha + \tfrac{z_\alpha^2 - 1}{6} \, \gamma_S \right), \qquad \mathrm{SCR}_\alpha \approx \sqrt{S_V} \left(z_\alpha + \tfrac{z_\alpha^2 - 1}{6} \, \gamma_S \right). \tag{26}$$

First-order differentials and gradient (production form)

For any scalar functional f, write $\mathrm{d}f = \langle \nabla_{\boldsymbol{n}} f, \mathrm{d}\boldsymbol{n} \rangle + \langle \nabla_{\boldsymbol{o}} f, \mathrm{d}\boldsymbol{o} \rangle + \langle \nabla_{\boldsymbol{x}} f, \mathrm{d}\boldsymbol{x} \rangle + \langle \nabla_{\boldsymbol{a}} f, \mathrm{d}\boldsymbol{a} \rangle$ with the Euclidean inner product.

Building blocks. From (23) and (24),

$$\nabla_{\boldsymbol{n}} S_V = \boldsymbol{x}^{\circ 2} \odot (\boldsymbol{a} + \boldsymbol{o}), \tag{27}$$

$$\nabla_{\boldsymbol{x}} S_V = 2\,\boldsymbol{n} \odot \boldsymbol{x} \odot (\boldsymbol{a} + \boldsymbol{o}), \tag{28}$$

$$\nabla_{\boldsymbol{a}} S_V = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 2}, \tag{29}$$

$$\nabla_{\boldsymbol{o}} S_V = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 2}, \tag{30}$$

$$F(\boldsymbol{a}, \boldsymbol{o}) := \boldsymbol{a}^{\circ 3} + 3\boldsymbol{a}^{\circ 2} + 3\boldsymbol{o} \odot \boldsymbol{a} + 2\boldsymbol{o}^{\circ 2} - \boldsymbol{o}, \tag{31}$$

$$\nabla_{\boldsymbol{n}} \mu_3 = \boldsymbol{x}^{\circ 3} \odot F, \tag{32}$$

$$\nabla_{\boldsymbol{x}} \mu_3 = 3\,\boldsymbol{n} \odot \boldsymbol{x}^{\circ 2} \odot F, \tag{33}$$

$$\nabla_{\boldsymbol{a}} \mu_3 = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 3} \odot \Big(3\boldsymbol{a}^{\circ 2} + 6\boldsymbol{a} + 3\boldsymbol{o} \Big), \tag{34}$$

$$\nabla_{\boldsymbol{o}} \mu_3 = \boldsymbol{n} \odot \boldsymbol{x}^{\circ 3} \odot \Big(3\boldsymbol{a} + 4\boldsymbol{o} - \mathbf{1} \Big). \tag{35}$$

Skewness differential. With $\gamma_S = \mu_3 / S_V^{3/2}$,

$$\mathrm{d}\gamma_S = \frac{\mathrm{d}\mu_3}{S_V^{3/2}} - \frac{3}{2} \frac{\mu_3}{S_V^{5/2}} \mathrm{d}S_V = \frac{\mathrm{d}\mu_3}{S_V^{3/2}} - \frac{3}{2} \frac{\gamma_S}{S_V} \mathrm{d}S_V. \tag{36}$$

SCR differential and gradient. Let

$$\alpha_V := \frac{1}{2\sqrt{S_V}} \Big(z_\alpha + \tfrac{z_\alpha^2 - 1}{6} \gamma_S \Big) - \frac{z_\alpha^2 - 1}{4} \frac{\gamma_S}{\sqrt{S_V}}, \qquad \beta_3 := \frac{z_\alpha^2 - 1}{6} \, \frac{1}{S_V}. \tag{37}$$

From (26) and (36),

$$\boxed{\mathrm{dSCR}_\alpha = \alpha_V \, \mathrm{d}S_V + \beta_3 \, \mathrm{d}\mu_3.} \tag{38}$$

Therefore the component-wise gradients are, for $p \in \{\, \boldsymbol{n}, \boldsymbol{o}, \boldsymbol{x}, \boldsymbol{a} \,\}$,

$$\boxed{\nabla_p \mathrm{SCR}_\alpha = \alpha_V \, \nabla_p S_V + \beta_3 \, \nabla_p \mu_3.} \tag{39}$$

Explicitly,

$$\nabla_{\boldsymbol{n}} \mathrm{SCR}_\alpha = \alpha_V \left(\boldsymbol{x}^{\circ 2} \odot (\boldsymbol{a} + \boldsymbol{o}) \right) + \beta_3 \left(\boldsymbol{x}^{\circ 3} \odot F \right), \tag{40}$$

$$\nabla_{\boldsymbol{x}} \mathrm{SCR}_\alpha = \alpha_V \left(2\,\boldsymbol{n} \odot \boldsymbol{x} \odot (\boldsymbol{a} + \boldsymbol{o}) \right) + \beta_3 \left(3\,\boldsymbol{n} \odot \boldsymbol{x}^{\circ 2} \odot F \right), \tag{41}$$

$$\nabla_{\boldsymbol{a}} \mathrm{SCR}_\alpha = \alpha_V \left(\boldsymbol{n} \odot \boldsymbol{x}^{\circ 2} \right) + \beta_3 \left(\boldsymbol{n} \odot \boldsymbol{x}^{\circ 3} \odot (3\boldsymbol{a}^{\circ 2} + 6\boldsymbol{a} + 3\boldsymbol{o}) \right), \tag{42}$$

$$\nabla_{\boldsymbol{o}} \mathrm{SCR}_\alpha = \alpha_V \left(\boldsymbol{n} \odot \boldsymbol{x}^{\circ 2} \right) + \beta_3 \left(\boldsymbol{n} \odot \boldsymbol{x}^{\circ 3} \odot (3\boldsymbol{a} + 4\boldsymbol{o} - \mathbf{1}) \right). \tag{43}$$

These yield the first-order total differential in pure vector form:

$$\mathrm{dSCR}_\alpha = \langle \nabla_{\boldsymbol{n}} \mathrm{SCR}_\alpha, \mathrm{d}\boldsymbol{n} \rangle + \langle \nabla_{\boldsymbol{o}} \mathrm{SCR}_\alpha, \mathrm{d}\boldsymbol{o} \rangle + \langle \nabla_{\boldsymbol{x}} \mathrm{SCR}_\alpha, \mathrm{d}\boldsymbol{x} \rangle + \langle \nabla_{\boldsymbol{a}} \mathrm{SCR}_\alpha, \mathrm{d}\boldsymbol{a} \rangle. \tag{44}$$

6 Single sub-LoB: compact one-line sensitivities

For (21), write $T = \sqrt{N_E(A + \mathrm{OD}_N)}$ and $\mathrm{CF} = z_\alpha + \frac{z_\alpha^2 - 1}{6}\gamma_S$. Then

$$\mathrm{SCR} = X_E \, T \, \mathrm{CF}, \qquad \frac{\partial \mathrm{SCR}}{\partial X_E} = T \, \mathrm{CF},$$

and with $E_A = 3A^2 + 6A + 3\,\mathrm{OD}_N$, $E_{OD} = 3A + 4\,\mathrm{OD}_N - 1$,

$$\frac{\partial \mathrm{SCR}}{\partial N_E} = \frac{X_E T}{2N_E} \left[\mathrm{CF} - \frac{z_\alpha^2 - 1}{6} \gamma_S \right], \tag{45}$$

$$\frac{\partial \mathrm{SCR}}{\partial A} = \frac{X_E T}{2(A + \mathrm{OD}_N)} \, \mathrm{CF} + \frac{X_E T}{6} (z_\alpha^2 - 1) \left[\frac{E_A}{\sqrt{N_E} (A + \mathrm{OD}_N)^{3/2}} - \frac{3\gamma_S}{2(A + \mathrm{OD}_N)} \right], \tag{46}$$

$$\frac{\partial \mathrm{SCR}}{\partial\, \mathrm{OD}_N} = \frac{X_E T}{2(A + \mathrm{OD}_N)} \, \mathrm{CF} + \frac{X_E T}{6} (z_\alpha^2 - 1) \left[\frac{E_{OD}}{\sqrt{N_E} (A + \mathrm{OD}_N)^{3/2}} - \frac{3\gamma_S}{2(A + \mathrm{OD}_N)} \right]. \tag{47}$$